Interband spectrum of weakly coupled stochastic lattice Ginzburg-Landau models

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We analyze the excitation spectrum of the generator associated with the relaxation rate to equilibrium in weakly coupled stochastic Ginzburg-Landau models on a spatial lattice $\mathbb{Z}^d$. The spectrum has a quasiparticle interpretation. Depending on d and on the specific interaction, by solving the Bethe-Salpeter equation in the ladder approximation, we show the existence of a stable particle above the upper envelope of the two-particle band, possessing a concave dispersion curve. This result furthers our knowledge about the spectrum of the stochastic dynamics generator.

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I. INTRODUCTION

We analyze the excitation spectrum of the generator associated with the relaxation rate to equilibrium in weakly coupled stochastic lattice Ginzburg-Landau (GL) models. Such models can be used to describe the evolution of an order parameter in statistical mechanical systems [1,2]. Recently, much attention has been paid to understand the quasiparticle structure of the stochastic dynamics generator in these models [3–7]. In these works, the picture of a massive particle and a bound state below the two-particle threshold (the bottom of the first band) was established, depending on the space dimension d, and on specific conditions on the interaction. In the present report, we show the existence of a stable particle above the top of the first band, with a concave dispersion for small momentum.

II. MODEL AND RESULTS

We consider a lattice of unbounded real continuous spin variables $\varphi(\vec{x})$, $\vec{x}\in\mathbb{Z}^d$. For time $t\in\mathbb{R}$, a stochastic dynamics is introduced by a Langevin type equation

$$\frac{\partial\varphi}{\partial t}(t,\vec{x})=-\frac{1}{2}\frac{\delta}{\delta\varphi(t,\vec{x})}A(\varphi(t,\vec{x}))+\eta(t,\vec{x}),$$

$$\varphi(\vec{x},0)=\psi(\vec{x}), \tag{1}$$

where $\psi(\vec{x})$ is some initial condition, A is the system action and $\{\eta(t,\vec{x})\}$, $\vec{x}\in\mathbb{Z}^d$, $t\in[0,\infty)$ is a family of Gaussian white-noise processes with expectations $E(\eta(t,\vec{x}))=0$ and $E(\eta(t,\vec{x})\eta(t',\vec{y}))=\delta(t-t')\delta_{\vec{x},\vec{y}}$. The action is of the GL type, i.e.,

$$A(\varphi(\vec{x}))=\sum_{\vec{x}\in\mathbb{Z}^d}\left\{\frac{1}{2}\left[\sum_{i=1}^{d}(\varphi(\vec{x}+\vec{e}_i)-\varphi(\vec{x}))^2+m^2\varphi(\vec{x})^2\right]+\lambda\mathcal{P}(\varphi(\vec{x}))\right\}, \tag{2}$$

$\vec{e}_i$ is the unit vector along the ith coordinate; $\mathcal{P}$ is an even polynomial of degree $2N$, bounded from below and starting with a quartic term; $m>0$ and $\lambda\geqslant 0$.

The dynamics introduced by Eq. (1) for the Markov process $\varphi(t)=\varphi(t,\vec{x})$ is associated with a Markov semigroup and leaves invariant the Gibbs probability distribution $d\mu=e^{-A(\varphi)}d\varphi/normalization$ defined by the action (2). Precisely, the time evolution of any function f of the spin configuration $\varphi(\vec{x})$ is given by $f_t(\psi)=E(f[\varphi(t)])$, with $\varphi(t=0)=\psi(\vec{x})$. It follows that f_t is determined by the Markov semigroup $\exp(-tB)$ with generator B, for $f=f(\{\varphi(\vec{x})\})$, given by

$$Bf=-\frac{1}{2}\sum_{\vec{x}\in\mathbb{Z}^d}\left[\frac{\delta^2}{\delta\varphi(\vec{x})^2}f-\frac{\delta A}{\delta\varphi(\vec{x})}\frac{\delta f}{\delta\varphi(\vec{x})}\right].$$

The spectrum of B is related to decay rates of correlation functions of an imaginary time quantum field theory through a Feynman-Kac formula (see [6,7]). Rather than analyze the spectrum of B, it is suitable to consider the unitarily equivalent Schrödinger Hamiltonian

$$\begin{aligned}H&=-\frac{1}{2}\sum_{\vec{x}\in\mathbb{Z}^d}\frac{\delta^2}{\delta\varphi(\vec{x})^2}+\frac{1}{4}\sum_{\vec{x}\in\mathbb{Z}^d}\left[\frac{1}{2}\left(\frac{\delta A}{\delta\varphi(\vec{x})}\right)^2-\frac{\delta^2 A}{\delta\varphi(\vec{x})^2}\right]\\&=-\frac{1}{2}\sum_{\vec{x}\in\mathbb{Z}^d}\frac{\delta^2}{\delta\varphi(\vec{x})^2}+\frac{1}{8}\sum_{\vec{x}\in\mathbb{Z}^d}\varphi(\vec{x})[(-\Delta+m^2)^2\varphi](\vec{x})\\&\quad+\frac{\lambda}{4}\sum_{\vec{x}\in\mathbb{Z}^d}[(-\Delta+m^2)\varphi](\vec{x})\mathcal{P}'(\varphi(\vec{x}))\\&\quad+\sum_{\vec{x}\in\mathbb{Z}^d}\left[\frac{\lambda^2}{8}\mathcal{P}'(\varphi(\vec{x}))^2-\frac{\lambda}{4}\mathcal{P}''(\varphi(\vec{x}))-\frac{(2d+m^2)}{4}\right].\end{aligned}$$

Here, $-\Delta$ is the Laplacian $(-\Delta\varphi)(\vec{x})=2d\varphi(\vec{x})-\Sigma_{|\vec{x}-\vec{y}|=1}\varphi(\vec{y})$. The lattice translation operator $T(\vec{x})=\exp[-i\vec{P}\cdot\vec{x}]$ commutes with H, which is additively renormalized so that its spectrum is positive and starts at zero. We analyze the joint spectrum of H, $\vec{P}$, where $\vec{P}$ is the momentum.

The above infinite-lattice formulas are formal. In [7], it was rigorously shown how to define them starting from the

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finite-lattice finite-time model and then taking the thermodynamic limit using a cluster expansion.

For $\lambda=0$, we have a quasiparticle interpretation for the spectrum. Spectral points are denoted by $(E,\vec{p})$, $E\geqslant 0$ and $\vec{p}\in[-\pi,\pi]^d$, and $(E,\vec{0})$ is referred to as a mass. In addition to the vacuum $(0,\vec{0})$, we have a quasiparticle energy-momentum (e-m) spectrum $(E_0(\vec{p})\equiv E_{\lambda=0}(\vec{p}),\vec{p})$, where $E_0(\vec{p})$ is the isolated dispersion curve $E_0(\vec{p})=\Sigma^d_{i=1}(1-\cos p^i)+m^2/2$, with mass $m^2/2$. The rest of the spectrum is given by $\cup_{n\geqslant 2}(E_{0,n}(\vec{p}),\vec{p})$ with $E_{0,n}(\vec{p})=\Sigma_{j=1}^n\ E_0(\vec{p}_j)$, $\Sigma_{j=1}^n\ \vec{p}_j=\vec{p}$, i.e., the energy-momentum spectrum of n noninteracting quasiparticles with total momentum $\vec{p}$. For $n=2$, the spectrum corresponds to a band. For instance, for $d=1$, the band has lower envelope $E^{\downarrow}_{0,2}(\vec{p})=4\sin^2(p/4)+m^2$, and upper envelope $E^{\uparrow}_{0,2}(\vec{p})=4\cos^2(p/4)+m^2$. For sufficiently large m and $n\geqslant 3$, and d, there is also band spectrum but, for sufficiently large n, the bands overlap.

To see what is known for $\lambda>0$, we first observe that the imaginary-time quantum field theory associated with the model is Gaussian for $\lambda=0$, and has covariance $[-d^2/dt^2+(-\Delta+m^2)^2]^{-1}$. As in [6,7], the G-L interaction in Eq. (2) is taken as $\mathcal{P}(\varphi)=\Sigma_{n=2}^N[a_n/(2n)!]:\varphi^{2n}:$, with $a_N>0$ and $:\ :$ meaning the Wick order with respect to the above covariance. With this, first within the ladder approximation in [6] and then analyzing the complete model in [7], if m is fixed large and λ chosen sufficiently small, it is shown that a quasiparticle persists with dispersion curve $E_\lambda(\vec{p})\geqslant E_\lambda(\vec{0})=m^2/2+\mathcal{O}(\lambda^2)$. The mass spectrum up to the two-particle threshold mass $2E_\lambda(\vec{0})$ was also determined. More precisely, for $d=1$, 2 and if $a_2<0$, there is a single point M_b in the mass spectrum interval $I_\lambda=(E_\lambda(\vec{0}),2E_\lambda(\vec{0}))$, located near $2E_\lambda(\vec{0})$. The bound state is absent and there is no mass spectrum in I_λ if $a_2>0$, for any d, or for $a_2<0$ and $d\geqslant 3$.

Here, we consider the mass spectrum in the region between the first and the second bands. In the ladder approximation, and for $a_2>0$ and $d=1$, 2, we find that there is mass spectrum M_a above and close to the mass of the upper envelope of the first band. Moreover, for small $|\vec{p}|$, we show that the associated dispersion curve for this interband state is concave while that of the bound state below the first band is convex. The results are depicted in Fig. 1.

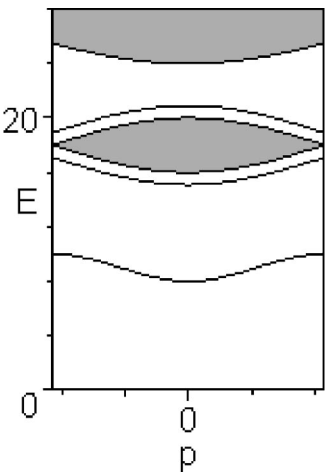

FIG. 1. The approximate e-m spectrum for the case $d=1$ and $m^2=16$. For $a_2<0$, only the isolated bound-state lower-dispersion curve appears; for $a_2>0$, only the isolated upper curve appears.

III. SPECTRAL ANALYSIS

To obtain the above results, we recall from [6,7] that the e-m spectrum associated with the two-particle states occurs as a k^0 singularity in the Fourier transform of a partially truncated four-point correlation expressed in terms of mixed relative coordinates (relative temporal center-of-mass coordinates and relative spatial coordinates) $\tilde{D}_\lambda(p,q,k)$, where $k=(k^0,\vec{k})$, $k^0\in\mathbb{R}$, $\vec{k}\in\mathbf{T}^d$, the d-dimensional torus. D_λ is the solution of the Bethe-Salpeter (BS) equation

$$D_\lambda=D^0_\lambda+D^0_\lambda K_\lambda D_\lambda,$$

where, letting $S_\lambda(\cdot,\cdot)$ denote the two-point function for the interacting model, $D^0_\lambda(x_1,x_2,x_3,x_4)=S_\lambda(x_1,x_3)S_\lambda(x_2,x_4)+S_\lambda(x_1,x_4)S_\lambda(x_2,x_3)$ and $D_\lambda(x_1,x_2,x_3,x_4)=S_\lambda(x_1,x_2,x_3,x_4)-S_\lambda(x_1,x_2)S_\lambda(x_3,x_4)$, and $S_\lambda(x_1,x_2,x_3,x_4)$ is the four-point function.

The action of $\tilde{D}_\lambda$ on functions $f(p)$, with $\int dq$ meaning $\int_0^\infty dq^0\int_{T^d}d\vec{q}$, is given by

$$(\tilde{D}_\lambda(k)f)(p)=\int\tilde{D}_\lambda(p,q,k)f(q)dq$$

and similarly for $\tilde{D}^0_\lambda$ and $\tilde{K}_\lambda$.

The determination of the spectrum in [6,7] above the one-particle state and below the two-particle threshold is restricted to total momentum $\vec{k}=\vec{0}$, i.e., only to mass spectrum. $K_\lambda=(D^0_\lambda)^{-1}-(D_\lambda)^{-1}$ is the B-S kernel and we write $K_\lambda=\lambda L+\lambda^2K^{(2)}$, where λL is called the ladder approximation to K_λ. Using the symmetries of K_λ under interchange of variables, $\tilde{L}_\lambda$ is calculated to be $\tilde{L}(p,q,k)=-(3/4)a_2[E_0(\vec{p})+E_0(\vec{q})+E_0(\vec{p}-\vec{k})+E_0(\vec{q}-\vec{k})]$, and has rank two. For $f(p)$ depending only on $\vec{p}$, we have

$$(\tilde{D}^0_\lambda(k)f)(p)=(2\pi)^{d+1}\bigg[\tilde{S}_\lambda\left(\frac{k^0}{2}-p^0,\ \vec{p}\right)\times\tilde{S}_\lambda\left(\frac{k^0}{2}+p^0,\vec{k}-\vec{p}\right)f(\vec{k}-\vec{p})+\tilde{S}_\lambda\left(\frac{k^0}{2}+p^0,\vec{p}\right)\times\tilde{S}_\lambda\left(\frac{k^0}{2}-p^0,\vec{k}-\vec{p}\right)f(\vec{p})\bigg]$$

and

$$(\tilde{L}_\lambda(k)\tilde{D}^0_\lambda(k)f)(p)$$

$$=-\frac{3a_2}{4}(2\pi)^{d+1}\left[\mathcal{E}_0(\vec{p},\vec{k})\int G(\vec{u},k)f(\vec{u})d\vec{u}+\int \mathcal{E}_0(\vec{u},\vec{k})G(\vec{u},k)f(\vec{u})d\vec{u}\right],$$

where $\mathcal{E}_0(\vec{u},\vec{k})=(1/2)[E_0(\vec{u})+E_0(\vec{u}-\vec{k})]$ and

$$G(\vec{u},k)=\int \tilde{S}_\lambda\left(\frac{k^0}{2}+u^0,\vec{u}\right)\tilde{S}_\lambda\left(\frac{k^0}{2}-u^0,\vec{k}-\vec{u}\right)du^0. \quad (3)$$

Thus, we can write

$$(f,\tilde{D}_\lambda(k)f)=2(2\pi)^{d+1}\int \overline{f}(\vec{p})G(\vec{p},k)g(\vec{p},k)d\vec{p}, \quad (4)$$

where $g(\cdot,k)=[1-(2\pi)^{-2(d+1)}\lambda\tilde{L}(k)\tilde{D}^0_\lambda(k)]^{-1}f(\cdot)$.

Now, recalling from [6,7] that k^0 singularities of Eq. (4) are points in the e-m spectrum, we determine the k^0 singularities of $G(\vec{p},k)$ and $g(\vec{p},k)$, for $\mathrm{Im}\,k^0\in(0,3m^2/2)$. From [6,7], $\tilde{S}_\lambda(p)$ has the representation

$$\tilde{S}_\lambda(p)=\frac{c_\lambda(\vec{p})}{(p^0)^2+E_\lambda(\vec{p})^2}+\int_{m^2}^{\infty}\frac{2E}{(p^0)^2+E^2}d\eta'_\lambda(E,\vec{p}),$$

where $c_\lambda(\vec{p})=1+\mathcal{O}(\lambda^2)$ and $d\eta'$ has support on odd states with more than one particle. Then, performing the p^0 integration in Eq. (3), we obtain

$$G(\vec{p},k)=\pi c_\lambda(\vec{p})c_\lambda(\vec{k}-\vec{p})\frac{E_\lambda(\vec{p})+E_\lambda(\vec{k}-\vec{p})}{E_\lambda(\vec{p})E_\lambda(\vec{k}-\vec{p})}\times\frac{1}{(k^0)^2+[E_\lambda(\vec{p})+E_\lambda(\vec{k}-\vec{p})]^2}+G_1(\vec{p},k), \quad (5)$$

where $G_1(\vec{p},k)$ is analytic in $\mathrm{Im}\,k^0\in(0,3m^2/2)$. From Eq. (5), we see that the k^0 singularity due to $G(\vec{p},k)$ in Eq. (4) is the first band.

Concerning $g(\cdot,k)$, the k^0 singularities come from the zeroes of $1-\mu_\pm(k)$ where $\mu_\pm(k)$ are the eigenvalues of $(2\pi)^{-(d+1)}\lambda\tilde{L}(k)\tilde{D}^0(k)$ on the space of functions generated by constants and $\mathcal{E}(\vec{p},k)$. $\mu_\pm(k)$ are found to be

$$\mu_\pm(k)=-3a_2(2\pi)^{-(d+1)}\lambda[\alpha(k)\pm(\beta(k)\gamma(k))^{1/2}],$$

where

$$\{\alpha(k),\beta(k),\gamma(k)\}=\int G(\vec{q},k)\{\mathcal{E}_0(\vec{q},k),1,[\mathcal{E}_0(\vec{q},k)]^2\}d\vec{q}.$$

We now take k^0 on the positive imaginary axis and let it approach the lower envelope of the band from below (α, β, and γ are positive) to obtain the bound-state mass M_b for $\mu_+(k)=1$, for $a_2<0$ and $d=1$, 2. Letting it approach the upper envelope from above (α, β, and γ are now negative), $\mu_-(k)=1$ gives us the mass M_a for $a_2>0$ and $d=1$, 2. For $d\geqslant 3$, there are no mass spectral points above or below the first band.

To determine the behavior of the bound-state mass dispersion curve near $\vec{k}=\vec{0}$, we define $F(\chi,\vec{k})=\mu_+(k^0=i\chi,\vec{k})-1$. Thus, $F(M_b,\vec{k})=0$, and the dispersion curve $\chi(\vec{k})$ satisfies $F(\chi(\vec{k}),\vec{k})=0$. Calculating $d\chi(\vec{k})/dk^j=[\partial F/\partial k^j]/[\partial F/\partial\chi]$, $j=1,\ldots,d$, a detailed analysis shows that $d\chi(\vec{k})/dk^j$ is positive for k^j positive and small. Smoothness and the fact that $(d\chi/dk^j)(\vec{k}=\vec{0})=0$ imply that $\chi(\vec{k})$ is convex for small $|\vec{k}|$. A similar analysis leads to the concavity of the dispersion curve associated with M_a, for small $|\vec{k}|$.

IV. CONCLUDING REMARKS

We have determined the interband e-m spectrum for dynamic stochastic lattice Landau-Ginzburg models with small polynomial interaction with equilibrium states in the single-phase region. The determination of the spectrum in models with equilibrium states in the multiphase region, critical models, and models in the large-noise regime is of interest.

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